

# The Wess-Zumino-Witten term in non-commutative two-dimensional fermion models

E.F. Moreno\* and F.A. Schaposnik<sup>†</sup>

*Departamento de Física, Universidad Nacional de La Plata*

*C.C. 67, (1900) La Plata, Argentina*

## Abstract

We study the effective action associated to the Dirac operator in two dimensional non-commutative Field Theory. Starting from the axial anomaly, we compute the determinant of the Dirac operator and we find that even in the  $U(1)$  theory, a Wess-Zumino-Witten like term arises.

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\*Investigador CONICET

<sup>†</sup>Investigador CICBA

# 1 Introduction

Interest in non-commutative spaces has been renewed after the discovery that non-commutative gauge theories naturally arise when D-branes with constant B fields are considered [1]-[2]. These works as well as that in [3] prompted many investigations both in field theory and in string theory (see references in [3]). Concerning gauge field theories, recent results on chiral and gauge anomalies [4]-[6] have shown that well-known results on “ordinary” models extend naturally and interestingly to the case in which non-commutative spaces are considered. In this work we consider a problem which can be seen as closely related to that of anomalies, namely the evaluation of the two-dimensional fermion determinant in non-commutative space-time. This problem is of interest not only for the analysis of two-dimensional QED and QCD in non-commutative space, but also in connection with abelian and non-Abelian bosonization since, as it is well-known, the knowledge of the fermion determinant leads more or less directly to the bosonization rules.

We start by evaluating in Section II the chiral anomaly in two-dimensional non-commutative space-time in a way adapted to the calculation of fermion determinants through integration of the anomaly. This last is done in Section III where both the Abelian and ( $U(N)$ ) non-Abelian fermion determinant is calculated exactly. In both cases we obtain for the determinant a Wess-Zumino-Witten term. Consequences of our results and possible extensions are discussed in section IV.

## 2 The chiral anomaly

### Conventions

As usual, we define the  $*$ -product between a pair of functions  $\phi(x)$ ,  $\chi(x)$  as

$$\begin{aligned}\phi * \chi(x) &\equiv \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\right)\phi(x)\chi(y)|_{x=y} \\ &= \phi(x)\chi(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu\phi\partial_\nu\chi(x) + O(\theta^2),\end{aligned}\tag{1}$$

and the (Moyal) bracket in the form

$$\{\phi, \chi\}(x) \equiv \phi(x) * \chi(x) - \chi(x) * \phi(x),\tag{2}$$

so that, when applied to (Euclidean) space-time coordinates  $x^\mu, x^\nu$ , one has

$$\{x^\mu, x^\nu\} = i\theta^{\mu\nu} \quad (3)$$

which is why one refers to non-commutative spaces. Here  $\theta^{\mu\nu}$  is a real, anti-symmetric constant tensor. Since we shall be interested in two dimensional space-time, one necessarily has  $\theta^{\mu\nu} = \theta \varepsilon^{\mu\nu}$  with  $\varepsilon^{\mu\nu}$  the completely anti-symmetric tensor and  $\theta$  a real constant. In the context of string theory, non-commutative spaces are believed to be relevant to the quantization of D-branes in background Neveu-Schwarz constant B-field  $B_{\mu\nu}$  [1]-[3]. In this context  $\theta^{\mu\nu}$  is related to the inverse of  $B^{\mu\nu}$ . Afterwards, this original interest was extended to the analysis of field theories in non-commutative space and then, as signaled in [6] it becomes relevant to know to what extent old problems and solutions in standard field theory fit in the new non-commutative framework.

A “non-commutative gauge theory” is defined just by using the  $*$ -product each time the gauge fields have to be multiplied. Then, even in the  $U(1)$  Abelian case, the curvature  $F_{\mu\nu}$  has a non-linear term (with the same origin as the usual commutator in non-Abelian gauge theories in ordinary space)

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ie (A_\mu * A_\nu - A_\nu * A_\mu) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ie \{A_\mu, A_\nu\} . \end{aligned} \quad (4)$$

This field strength is gauge-covariant (not gauge-invariant, even in the Abelian case) under gauge transformations which should be represented by  $U$  elements of the form

$$U(x) = \exp_*(i\lambda) \equiv 1 + i\lambda - \frac{1}{2}\lambda * \lambda + \dots \quad (5)$$

The covariant derivative implementing infinitesimal gauge transformations takes the form

$$\mathcal{D}_\mu[A]\lambda = \partial_\mu \lambda + ie (\lambda * A_\mu - A_\mu * \lambda) \quad (6)$$

so that an infinitesimal gauge transformation on  $A_\mu$  reads as usual

$$\delta A_\mu = \frac{1}{e} \mathcal{D}_\mu \lambda \quad (7)$$

Concerning finite gauge transformations, one has

$$A_\mu^U = \frac{i}{e} U(x) * \partial_\mu U^{-1}(x) + U(x) * A_\mu * U^{-1}(x) \quad (8)$$

Given a fermion field  $\psi$ , one can easily see that the combination

$$\gamma^\mu D_\mu[A]\psi = \gamma^\mu \partial_\mu \psi - ie\gamma^\mu A_\mu * \psi \quad (9)$$

transforms covariantly under gauge transformations (8),

$$\gamma^\mu D_\mu[A^U]\psi^U = U * \gamma^\mu D_\mu[A]\psi \quad (10)$$

with

$$\psi^U = U(x) * \psi \quad (11)$$

and

$$U(x) * U^{-1}(x) = U^{-1} * U(x) = 1 \quad (12)$$

A gauge invariant Dirac action can be defined in the form

$$S_f = \int d^d x \bar{\psi}(x) * i\gamma^\mu D_\mu[A]\psi(x) \quad (13)$$

## The Anomaly

Chiral transformations will be written as

$$\psi'(x) = U_5(x) * \psi \quad (14)$$

with

$$U_5(x) = \exp_*(\gamma_5 \alpha(x)) = 1 + \gamma_5 \alpha + \frac{1}{2} \alpha(x) * \alpha(x) + \dots \quad (15)$$

The chiral anomaly  $\mathcal{A}_d$  in  $d$ -dimensional space can be calculated from the formula

$$\log J_d[\alpha] = -2\mathbf{A}_d, \quad (16)$$

$$\mathbf{A}_d = \text{Tr } \gamma_5 \delta\alpha(x)|_{reg} \quad (17)$$

here  $J_d[\alpha]$  is the Fujikawa Jacobian associated with an infinitesimal chiral transformation  $U = 1 + \gamma_5 \delta\alpha$  and Tr includes a matrix and functional space trace.

Let us specialize to the two dimensional case. We shall use the heat-kernel regularization so that (17) will be understood as

$$\mathbf{A}_2 = \int d^2 x \mathcal{A}_2(x) * \delta\alpha(x), \quad (18)$$

$$\mathcal{A}_2(x) = \lim_{M \rightarrow \infty} \text{Tr} \gamma_5 \exp_* \left( \frac{\not{D} * \not{D}}{M^2} \right) . \quad (19)$$

After some standard manipulations, (19) takes the form

$$\mathcal{A}_2(x) = \frac{1}{4\pi} \text{tr} \gamma_5 \not{D} * \not{D} = \frac{1}{4\pi} \text{tr} (\gamma_5 \gamma^\mu \gamma^\nu) D_\mu * D_\nu . \quad (20)$$

Here  $\text{tr}$  is just the matrix trace. Using  $\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu) = 2i \varepsilon^{\mu\nu}$ , eq.(20) can be written as

$$\mathcal{A}_2(x) = \frac{e}{2\pi} \varepsilon^{\mu\nu} (\partial_\mu A_\nu - ie A_\mu * A_\nu) = \frac{e}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} . \quad (21)$$

This result coincides with that first obtained in [4].

### 3 The two-dimensional fermion determinant

Let us write the gauge field in the two-dimensional case in the form

$$\not{A} = \frac{1}{e} (i \not{\partial} \exp_* (\gamma_5 \phi + i\eta)) * \exp_* (-\gamma_5 \phi - i\eta) \quad (22)$$

Note that in the  $\theta_{\mu\nu} \rightarrow 0$  limit, eq.(22) reduces to the usual decomposition of a two-dimensional gauge field in the form

$$e A_\mu = \varepsilon_{\mu\nu} \partial^\nu \phi + \partial_\mu \eta \quad (23)$$

which allows to decouple fermions from the gauge-field and then obtain the fermion determinant as the Jacobian associated to this decoupling [8]. Now, the form (22) was precisely proposed in [9] to achieve the decoupling in the case of non-Abelian gauge field backgrounds, this leading to the calculation of the  $QCD_2$  fermion determinant in a closed form. Afterwards [10], it was shown that writing a two dimensional gauge field as in eq.(22) (without the  $*$ -product but in the  $U(N)$  case) does correspond to the choice of a gauge condition . Eq.(22) is then the extension of this approach for a case in which non-commutativity arises from the use of the  $*$ -product.

At the classical level, the change of fermionic variables

$$\begin{aligned} \psi &= \exp_* (\gamma_5 \phi + i\eta) * \chi \\ \bar{\psi} &= \bar{\chi} * \exp_* (\gamma_5 \phi - i\eta) \end{aligned} \quad (24)$$

completely decouples the gauge field, written as in (22), leading to an action of free massless fermions,

$$S_f = \int d^2x \bar{\chi} * i\partial\chi \quad (25)$$

Of course, this is not the whole story: at the quantum level there is a Fujikawa Jacobian  $J$  [7] associated to change (24). In order to compute this Jacobian, we follow the method introduced in [8]-[9]. Consider then the change of variables

$$\begin{aligned} \psi &= U_t * \chi_t, \\ \bar{\psi} &= \bar{\chi}_t * U_t^\dagger \end{aligned} \quad (26)$$

where

$$U_t = \exp_*(t(\gamma_5\phi + i\eta)) \quad , \quad (27)$$

and  $t$  is a real parameter,  $0 \leq t \leq 1$ . Given the fermion determinant defined as

$$\det(\partial - ie\mathcal{A}) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(-S_f[\bar{\psi}, \psi]) \quad (28)$$

we proceed to the change of variables (26) which leads to

$$\begin{aligned} \det(\partial - ie\mathcal{A}) &= J[\phi, \eta; t] \int \mathcal{D}\bar{\chi}_t \mathcal{D}\chi_t \exp(-S_f[\bar{\chi}_t, \chi_t]) \\ &= J[\phi, \eta; t] \det D_t \end{aligned} \quad (29)$$

where  $J[\phi, \eta; t]$  stands for the Jacobian

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = J[\phi, \eta; t] \mathcal{D}\bar{\chi}_t \mathcal{D}\chi_t \quad (30)$$

and we have defined

$$D_t = U_t^\dagger * (\partial - ie\mathcal{A}*) U_t \quad (31)$$

Now, since the l.h.s. in (29) does not depend on  $t$  we get, after differentiation,

$$\frac{d}{dt} \log \det D_t = -\frac{d}{dt} \log J[\phi, \eta; t] \quad (32)$$

or, after integrating on  $t$  and using that  $D_0 = \partial - ie\mathcal{A}$  and  $D_1 = \partial$

$$\det(\partial - ie\mathcal{A}) = \det \partial \exp\left(-2 \int_0^1 dt \mathbf{A}_2(t)\right) \quad (33)$$

where we have used

$$\mathbf{A}_2(t) = \frac{d}{dt} \log J[\phi, \eta; t] \quad (34)$$

Now, it is trivial to identify  $\mathbf{A}_2(t)$  with the two-dimensional chiral anomaly as defined in eq.(17), just by writing  $\delta\alpha = \phi dt$ ,

$$\mathbf{A}_2(t) = \text{Tr}(\gamma_5 * \phi)|_{reg} \quad (35)$$

In order to have a gauge-invariant regularization ensuring that the  $\eta$  part of the transformation does not generate a Jacobian, we adopt, in agreement with (18) and (19),

$$\mathbf{A}_2(t) = \lim_{M \rightarrow \infty} \text{Tr} \left( \gamma_5 \exp \left( \frac{\mathcal{D}_t^* \mathcal{D}_t}{M^2} \right) * \phi \right) \quad (36)$$

so that finally one has

$$\mathbf{A}_2(t) = \frac{e}{2\pi} \int d^2x \varepsilon^{\mu\nu} \left( \partial_\mu A_\nu^t - ie A_\mu^t * A_\nu^t \right) * \phi = \frac{e}{4\pi} \int d^2x \varepsilon^{\mu\nu} F_{\mu\nu}^t * \phi \quad (37)$$

where we have introduced

$$\gamma_\mu A_\mu^t = -\frac{1}{e} (i \not{\partial} U_t) * U_t^{-1} \quad (38)$$

and analogously for  $F_{\mu\nu}^t$ . In summary, we can write for the  $U(1)$  fermion determinant

$$\det(\not{\partial} - ie \not{A}) = \exp \left( -\frac{e}{2\pi} \int d^2x \int_0^1 dt \varepsilon^{\mu\nu} F_{\mu\nu}^t * \phi \right) \det \not{\partial} \quad (39)$$

It will be convenient to use the relation

$$\gamma^\mu \gamma_5 = -i \varepsilon^{\mu\nu} \gamma_\nu \quad (40)$$

to rewrite (39) in the form

$$\det(\not{\partial} - ie \not{A}) = \exp \left( \frac{ie}{2\pi} \text{tr} \int d^2x \int_0^1 dt \gamma_5 \phi * (\not{\partial} A^t - ie A^t * A^t) \right) \det \not{\partial} \quad (41)$$

Then, one can exploit the identity

$$\begin{aligned} \text{tr} \int d^2x \frac{1}{2} \frac{d}{dt} A^t * A^t &= \frac{1}{e} \text{tr} \int d^2x \gamma_5 i \not{\partial} \phi * A^t + 2 \text{tr} \int d^2x \gamma_5 A^t * \phi * A^t \\ &+ \frac{1}{e} \text{tr} \int d^2x (\not{\mathcal{D}} \eta) * A \end{aligned} \quad (42)$$

and find for (41)

$$\begin{aligned} \log \det(\not{\partial} - ie \not{A}) &= -\frac{e^2}{4\pi} \text{tr} \int d^2x \not{A} * \not{A} + \frac{e^2}{2\pi} \text{tr} \int dt \int d^2x \gamma_5 \phi * \not{A} * \not{A} \\ &+ \frac{e}{2\pi} \int dt \int d^2x (\not{D}\eta) * \not{A} + \log \det \not{\partial} \end{aligned} \quad (43)$$

This is the final form for the fermion determinant in a  $U(1)$  gauge theory. In order to write it in a more suggestive way connecting it with the Wess-Zumino-Witten term, let us consider the light-cone gauge  $A_+ = 0$ . Then, one can see after some algebra that [11]

$$\begin{aligned} \log \left( \frac{\det(\not{\partial} - ie \not{A})}{\det \not{\partial}} \right) &= -\frac{1}{8\pi} \int d^2x \left( \partial_\mu g(x)^{-1} \right) * (\partial_\mu g(x)) \\ &+ \frac{i}{12\pi} \epsilon_{ijk} \int_B d^3y g(x, t)^{-1} * (\partial_i g(x, t)) * g(x, t)^{-1} * (\partial_j g(x, t)) g^{-1} * (\partial_k g(x, t)) \end{aligned} \quad (44)$$

here we have written  $A_- = (i/e)g(x) * \partial_- g^{-1}(x)$  with  $g(x) = \exp_*(2\phi(x))$ ,  $g(x, t) = \exp_*(2\phi(x)t)$  and  $d^3y = d^2x dt$  so that the integral in the second line of eq.(44) runs over the three dimensional manifold  $B$ , which in compactified Euclidean space can be identified with a ball with boundary  $S^2$ . Index  $i$  runs from 1 to 3. As in the ordinary commutative case, because the determinant was computed in Euclidean space, elements  $g$  should be considered as belonging to  $U(1)_C$  (the complexified  $U(1)$ ) [11]-[12].

So, we have found for the two-dimensional non-commutative fermion determinant that, even for a  $U(1)$  gauge field background, a Wess-Zumino-Witten term arises due to non-commutativity of the  $*$ -product. Of course, in the  $\theta^{\mu\nu} \rightarrow 0$  limit in which the  $*$ -product becomes the ordinary one, the  $U(1)$  fermion determinant contribution to the gauge field effective action reduces to  $(-1/2\pi) \int d^2x \phi \partial^\mu \partial_\mu \phi$  which is nothing but the Schwinger determinant result expressed in a gauge-invariant way.

The method we have employed has the advantage that it can be trivially generalized to the case of a  $U(N)$  gauge group. One has just to take into account that in (22) one has

$$\phi = \phi^a t^a, \quad \eta = \eta^a t^a \quad (45)$$

with  $t^a$  the  $U(N)$  generators. Then, as originally shown in [9] for the commutative case, the fermion determinant can be seen to be given by

$$\det(\not{\partial} - ie \not{A}) = \exp \left( -\frac{e}{4\pi} \text{tr}^c \int d^2x \int_0^1 dt \varepsilon^{\mu\nu} F_{\mu\nu}^t * \phi \right) \det \not{\partial} \quad (46)$$



where  $\text{tr}^c$  is a trace over the  $U(N)$  algebra. Then, following the same steps leading to (44), one gets, in the  $U(N)$  case

$$\begin{aligned} \log \left( \frac{\det(\not{\partial} - ie \not{A})}{\det \not{\partial}} \right) &= -\frac{1}{8\pi} \text{tr}^c \int d^2x \left( \partial_\mu g^{-1} \right) * (\partial_\mu g) \\ &\quad + \frac{i}{12\pi} \epsilon_{ijk} \text{tr}^c \int_B d^3y g^{-1} * (\partial_i g) * g^{-1} * (\partial_j g) g^{-1} * (\partial_k g) \end{aligned} \quad (47)$$

where again, in the light-cone gauge we have written

$$A_- = -\frac{i}{e} g * \partial_- g^{-1}, \quad A_+ = 0 \quad (48)$$

$$g = \exp_*(2\phi^a t^a) \quad (49)$$

Eq. (47) is the generalization of the expression given in [13] for the two-dimensional non-Abelian fermion determinant to the case of non commutative space-time.

## 4 Conclusion

We studied in this article the effective action of the gauge degrees of freedom in a two dimensional non-commutative Field Theory of fermions coupled to a gauge field. Using Fujikawa's approach, we computed the chiral anomaly and, from it, the fermionic determinant of the non-commutative Dirac operator.

As it was to be expected, the result for the fermion determinant corresponds to the  $*$ -deformation of the standard result. Now, the fact that a Moyal bracket enters in the field strength curvature even in the Abelian case, has important consequences, some of which have already been signaled in [4]-[6] where chiral and gauge anomalies in non-commutative spaces have been analyzed.

In our framework, where the anomaly was integrated in order to obtain the fermion determinant, this reflects in the fact that a Wess-Zumino-Witten like term arises both in the Abelian and in the non-Abelian cases (eqs.(44) and (47) respectively). This should have, necessarily, implications in relevant aspects of two-dimensional theories since, as it is well-known, bosonization is closely related to the form of the fermion determinant [15]. Indeed, the bosonization rules for fermion currents as well as the resulting current algebra

can be easily derived by differentiation of the Dirac operator determinant  $\det(\not{d} - i \not{s})$  with respect to the source  $s_\mu$  (see [16] for a review). Now, as one learns from ordinary non-Abelian bosonization, where the Polyakov-Wiegmann identity plays a central rôle in the bosonization recipe, here one should have an analogous identity which will lead to non-trivial changes at the level of currents and, a fortiori, for the current algebra. In view of the relevance of these objects in connection with two-dimensional bosonic and fermionic models, it will be worthwhile to pursue the investigation initiated here in this direction.

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